SOME VERY NON-KÄHLER MANIFOLDS: THE FRÖLICHER SPECTRAL SEQUENCE CAN BE ARBITRARILY NON DEGENERATE

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Abstract. The Frölicher spectral sequence of a compact complex manifold X measures the difference between Dolbeault cohomology and de Rham cohomology. If X is Kähler then the spectral sequence collapses at the E_1 term and no example with $d_n \neq 0$ for $n > 3$ has been described in the literature.

We construct for $n \geq 2$ nilmanifolds with left-invariant complex structure X_n such that the n-th differential d_n does not vanish. This answers a question mentioned in the book of Griffiths and Harris.

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Introduction. Let X be a compact complex manifold. One of the most basic invariants of the holomorphic structure of X are its Hodge-numbers

$$h^{p,q}(X) = \dim H^{p,q}(X) = \dim H^q(X, \Omega^p_X)$$

and it is an important question how these are related to topological invariants like the Betti-numbers

$$b_k(X) = \dim H^k(X, \mathbb{C}).$$

In order to produce relations between these Frölicher studied in [Frö55] a spectral sequence connecting Dolbeault cohomology and de Rham cohomology: if we denote by $(\mathcal{A}^k(X), d)$ the complex valued de Rham complex then the decomposition of the exterior differential $d = \partial + \bar\partial$ gives rise to a decomposition

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

where $\mathcal{A}^{p,q}(X)$ is the space of forms of type (p, q). The resulting double complex $(\mathcal{A}^{p,q}(X), \partial, \bar\partial)$ yields a spectral sequence such that

$$\begin{aligned} E_0^{p,q} &= \mathcal{A}^{p,q}(X) \qquad d_0 = \bar\partial, \\ E_1^{p,q} &= H^{p,q}(X) \qquad d_1 = [\partial], \\ E_n^{*,*} &\Rightarrow H^*_{dR}(X, \mathbb{C}), \end{aligned}$$

the so-called Frölicher spectral sequence (see e.g. [GH78], p. 444). It is well known that it degenerates at the E_1 term for Kähler manifolds, which can for example be deduced from the Hodge-decomposition $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.

Therefore non-degenerating differentials in higher steps measure how much Dolbeault cohomology differs from de Rham cohomology and, in some sense, how far X is from being a Kähler manifold.

Kodaira [Kod64] showed that for compact complex surfaces d_1 is always zero and the first example with $d_1 \neq 0$ was the Iwasawa manifold: consider the nilpotent complex Lie group

$$G := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{C} \right\}$$

and the discrete cocompat subgroup $\Gamma := G \cap \mathrm{Gl}(3, \mathbb{Z}[i]) \subset G$. Then $X := G/\Gamma$ is a complex parallelisable nilmanifold such that $E_1 \ncong E_2 = E_\infty$. Later it was shown that for such manifold we have always $d_2 = 0$ [CFG91, Sak76].

For a long time no manifolds with $d_2 \neq 0$ were known and it was in fact speculated if $E_2 = E_\infty$ holds for every compact complex manifold.

Eventually some examples with $d_2 \neq 0$ were found independently by Cordero, Fernández and Gray [CFG87], who used a nilmanifold with left-invariant complex structure of complex dimension 4, and Pittie [Pit89], who gave a simply-connected example by constructing a left-invariant complex structure on Spin(9).

Cordero, Fernández and Gray continued their study in [CFG91] finding a complex 6-dimensional nilmanifold such that $E_3 \ncong E_4 = E_\infty$ and together with Ugarte they showed in [CFGU99] that for 3-folds several different non-degeneracy phenomena can occur up to $E_2 \ncong E_3 = E_\infty$.

The aim of this short note is to answer the question mentioned in the book of Griffiths and Harris [GH78] and repeated by Cordero, Fernández and Gray if we can exhibit manifolds with $d_n \neq 0$ for arbitrary large n.

Theorem 1 — *For every $n \geq 2$ there exist a complex $2n$-dimensional nilmanifold with left-invariant complex structure X_n such that the Frölicher spectral sequence does not degenerate at the E_n term, i.e., $d_n \neq 0$.*

We will present the example in an elementary way without going into the general theory of nilmanifolds with left-invariant complex structure.

Construction of the example. Consider for $n \geq 2$ the *real*, nilpotent subgroup of $\mathrm{Gl}(n+3, \mathbb{C})$

$$G_n := \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & -\bar{x}_{n-1} & x_n & y_n \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 1 & 0 & -\bar{x}_1 & x_2 & y_2 \\ & & & 1 & 0 & -\bar{x}_1 & y_1 \\ & & & & 1 & 0 & x_1 \\ \vdots & & & & \ddots & 1 & x_1 \\ 0 & \dots & & & \dots & 0 & 1 \end{pmatrix} : x_k, y_k \in \mathbb{C} \right\}. \tag{1}$$

Regarding $x_1, \dots, x_n, y_1, \dots, y_n$ as complex coordinates we can identify G_n with $\mathbb{C}^{2n}$ and then multiplication on the left with a fixed element is holomorphic. Taking the quotient with respect to the discrete subgroup $\Gamma := G_n \cap \mathrm{Gl}(n+3, \mathbb{Z}[i])$ acting on the left yields a (compact) nilmanifold with left-invariant complex structure $X_n := \Gamma \backslash G_n$.

If we call the matrix in (1) A then the space of left-invariant differential $(1,0)$-forms is spanned by the components of $A^{-1}dA$ which yields the following:

$$U := \{dx_1, \dots, dx_n, \omega_1, \dots, \omega_n\} \tag{2}$$

where

$$\begin{aligned} \omega_1 &= dy_1 + \bar{x}_1 dx_1, \\ \omega_k &= dy_k + (\bar{x}_{k-1} - x_k) dx_1 \qquad (k \geq 2). \end{aligned}$$

The differential of the above forms is readily calculated as

$$
\begin{aligned}
d(dx_k) &= 0 && (k = 1, \ldots, n),\\
d\omega_1 &= \bar\partial\omega_1 = d\bar x_1 \wedge dx_1, &&\\
d\omega_k &= \underbrace{-dx_k \wedge dx_1}_{=\partial\omega_k} \underbrace{-dx_1 \wedge d\bar x_{k-1}}_{=\bar\partial\omega_k} && (k \geq 2).
\end{aligned}
\tag{3}
$$

The claim of Theorem 1 follows now directly from

Lemma 2 — *The differential form $\beta_1 := \bar\omega_1 \wedge d\bar x_2 \wedge \cdots \wedge \bar x_{n-1}$ defines a class in $E_n^{0,n-1}$ and*

$$d_n([\beta_1]) = [dx_1 \wedge \cdots \wedge dx_n] \neq 0 \text{ in } E_n^{n,0}.$$

Proof. Following the exposition in [BT82] (§14, p.161ff) we say that an element $\beta_0 \in E_0^{p,q}$ lives to E_r if it represents a cohomology class in E_r or equivalently if it is a cocycle in $E_0, E_1, \ldots, E_{r-1}$. This is shown to be equivalent to the existence of a *zig-zag* of length r, that is, a collection of elements $\beta_1, \ldots, \beta_{r-1}$ such that

$$\beta_i \in E_0^{p+i,q-i}, \quad \bar\partial\beta_0 = 0, \quad \partial\beta_{i-1} + \bar\partial\beta_i = 0\,(i = 1, \ldots r-1).$$

These can be represented as

$$
\begin{tikzcd}
0 & & & & \\
\beta_1 \arrow[u, mapsto, "\bar\partial"] \arrow[r, mapsto, "\partial"] & {} & & & \\
& \beta_2 \arrow[u, mapsto] \arrow[r, mapsto] & {} & & \\
& & \ddots & {} & \\
& & & \beta_r \arrow[u, mapsto] \arrow[r, mapsto] & \partial\beta_r.
\end{tikzcd}
$$

In this picture we have the first quadrant double complex given by $(E_0^{p,q}, \partial, \bar\partial)$ in mind in which this zig-zag lives.

Furthermore $d_r([\beta_0]) = [\partial\beta_r]$ is zero in $E_r^{p+r,q-r+1}$ if and only if there exists an element $\beta_{r+1} \in E_0^{p+r,q-r}$ such that $\bar\partial\beta_r + \partial\beta_{r-1} = 0$, i.e., we can extend the zig-zag by one element.

We will now show that the left-invariant differential form β_1 admits a zig-zag of length n which cannot be extended.

For $k = 2, \ldots, n$ define

$$\beta_k := dx_2 \wedge \cdots \wedge dx_{k-1} \wedge \omega_k \wedge d\bar x_k \wedge \cdots \wedge d\bar x_{n-1}.$$

It is now a straight-foreward calculation to show that

$$
\begin{gathered}
\bar\partial\beta_1 = 0,\\
\partial\beta_n = dx_1 \wedge \cdots \wedge dx_n,\\
\partial\beta_1 = -\bar\partial\beta_2 = dx_1 \wedge d\bar x_1 \wedge \cdots \wedge d\bar x_{n-1},
\end{gathered}
$$

and for $2 \leq k \leq n-1$

$$\bar\partial\beta_{k+1} = -\partial\beta_k = (-1)^k dx_2 \wedge \cdots \wedge dx_k \wedge dx_1 \wedge d\bar x_k \wedge \cdots \wedge d\bar x_{n-1}.$$

Therefore these elements define a zig-zag such that

$$d_n[\beta_1] = [\partial\beta_n] = [dx_1 \wedge \cdots \wedge dx_n] \neq 0$$

which completes the proof. □

Remark 3 — The manifold X_n admits two simple geometric descriptions in terms of torus bundles: the centre of G_n is given by the matrices for which $x_1 = \ldots = x_n = 0$ and hence isomorphic (as a Lie group) to $\mathbb{C}^n$. This yields an exact sequence

$$0 \to \mathbb{C}^n \to G_n \to \mathbb{C}^n \to 0$$

which is compatible with the action of Γ and thus we get a principal holomorphic torus bundle $X_n \to T_n = \mathbb{C}^n/\mathbb{Z}[i]^n$ with fibres again isomorphic to T_n.

Considering instead the abelian normal subgroup $H \cong \mathbb{C}^{2n-1}$ generated by the matrices for which $x_1 = 0$ we get another sequence

$$0 \to H \to G_n \to \mathbb{C} \to 0$$

and can describe X_n also as a holomorphic torus bundle over a 1-dimensional complex torus. In this case we have no longer a principal bundle since the extension is not central, i.e., the structure group does not consist of translations.

Remark 4 — In order to construct the nilmanifold with left-invariant complex structure X_n we followed a standard procedure which can more in general be used to construct manifolds with exotic cohomological properties.

One observes that G_n is determined (up to canonical isomorphism) by the data given in (2) and (3): via the identity

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = -\alpha([X, Y])$$

for left-invariant differential forms and vector-fields we get both the Lie algebra structure on $\mathfrak{g}_n$, the tangent space at the identity, and the complex structure J on $\mathfrak{g}_n$ (setting ${\mathfrak{g}_n}^{0,1} = \operatorname{Ann} U$).

Our choice of rational structure constants also guarantees the existence of a cocompact discrete subgroup and hence we have reconstructed X_n (up to some regular covering).

The use of left-invariant differential forms in the proof is not a coincidence: Nomizu proved in [Nom54] that the de Rham cohomology of a nilmanifold can be described by left-invariant differential forms and, under some mild assumptions, which are verified for principal holomorphic torus bundles, this holds also for Dolbeault cohomology (see [CF01, CFGU99, Cat04]). In other words the whole cohomology algebra of X_n is incoded in (2) and (3).

More information on nilmanifolds with left-invariant complex structures can be found for example in [CFGU99, CF01, Sal01, Rol07].

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